Exam. Code : 211004
Subject Code : 4642

M.Sc. (Mathematics) 4th Semester
MATH-586 : NUMBER THEORY

Time Allowed—2 Hours] [Maximum Marks—100

Note :— Attempt any ***four*** questions. All questions carry equal marks.

1. (a) Does the system of congruences $2x \equiv 6 \pmod 7$, $3x \equiv 1 \pmod{35}$ have a simultaneous solution ? Justify.

   (b) State and prove Chinese Remainder Theorem.

   (c) Find all primitive roots modulo 11.

2. (a) If $m > 2$, $n > 2$ are such that g.c.d $(m, n) = 1$, prove that there are no primitive roots modulo mn.

   (b) If r is a primitive root modulo an odd prime p, prove that $r^{\frac{p-1}{2}} \equiv -1 \pmod p$.

   (c) Is product of two primitive roots modulo p is also a primitive root modulo p ? Justify.

3. (a) Prepare a table of indices for a primitive root of 13 and use it to solve the congruence $x^3 \equiv 4 \pmod{13}$.

   (b) Prove that for any odd prime p, half of the elements in $\{1, 2, ..., p-1\}$ are quadratic residues modulo p and the remaining half are quadratic non-residues modulo p.

   (c) Given a prime p, show that for some choice of $n > 0$, p divides $(n^2 - 2)(n^2 - 3)(n^2 - 6)$.

4. (a) State and prove quadratic reciprocity law.

(b) If p is an odd prime, prove that :

$$\left(\frac{2}{p}\right) = \begin{cases} 1 \text{ if } p \equiv \pm 1 \pmod 8 \\ -1 \text{ if } p \equiv \pm 3 \pmod 8 \end{cases}$$

5. (a) Prove that $\sigma(n)$ is an odd integer if and only if n is a perfect square or twice a perfect square.

(b) State and prove Mobius Inversion formula.

6. (a) If (x, y, z) is a Pythagorean triplet such that g.c.d (x, y, z) = 1. Prove that 60 divides xyz.

(b) Characterize all prime numbers that can be written as sum of two squares.

7. (a) Prove that every irrational number has a unique representation as an infinite continued fraction.

(b) Let x be an irrational number. If $\frac{a}{b}$ with $b \geq 1$ and g.c.d (a, b) = 1, is a rational number such that $\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$, prove that $\frac{a}{b}$ is one of the convergents of the continued fraction representation of x.

8. (a) Find smallest positive solution of the Pell's equation $x^2 - 13y^2 = 1$.

(b) Prove that any periodic simple continued fraction is a quadratic irrational number.